Prof. Amador Martin-Pizarro — Albert-Ludwigs-Universität Freiburg
Übungen: Xier Ren — Sommersemester 2024

Topology

Problem Sheet 8

Deadline: 18 June 2024, 15h

Exercise 1 (5 Points)**.**

Let $(X_i, \mathcal{T}_i)_{i\in I}$ be a family of topological spaces. such that the space $\prod_{i\in I} X_i$ equipped with the product topology is locally compact.

a) Show that every X_i is locally compact for i in I. Furthermore, show that for all but finitely many i's in I, the space X_i must be compact.

b) For which non-empty index set I is the product $\prod_{i\in I}(\mathbb{R}, \mathcal{T}_{eucl})$ locally compact?

Exercise 2 (7 Points)**.**

Consider $\mathbb{N}$ with the discrete topology.

a) Describe all compact subsets of $\mathbb{N}$.

b) Is $\mathbb{N}$ compact? Is it locally compact?

c) Is the map
$$f: \begin{array}{rcl} \mathbb{N} & \to & \mathbb{N} \\ n & \mapsto & \begin{cases} 0, & \text{for } n \text{ even} \\ 1, & \text{otherwise} \end{cases} \end{array}$$
continuous?

d) Is it possible to extend f to a continuous map $\tilde{f}: X \to X$ such that $\tilde{f}_{\restriction \mathbb{N}} = f$, where X is the one-point compactification of $\mathbb{N}$?

Exercise 3 (5 Points)**.**

For every natural number k in $\mathbb{N}$, consider the following loop on $\mathbb{S}^1$ based on $(1,0)$:

$$\beta_k : \begin{array}{rcl} [0,1] & \to & \mathbb{S}^1 \\ t & \mapsto & (\cos(2\pi i k t), \sin(2\pi i k t)) \end{array}$$

Produce an explicit homotopy between $\beta_k \star \beta_\ell$ and $\beta_{k+\ell}$.

Exercise 4 (3 Points)**.**

Given two topological spaces X and Y, the continuous maps $f, g : X \to Y$ are *homotopic* if there is a continuous map $H : X \times [0,1] \to Y$ (with respect to the product topology on $X \times [0,1]$) such that

$$H(x,0) = f(x) \text{ und } H(x,1) = g(x) \text{ für alle } x \in X.$$

If f and g are homotopic via the homotopy H with

$$f(x_0) = g(x_0) = y_0 \text{ and } H(x_0, t) = y_0,$$

show that their pushforwards $f_\star$ and $g_\star$ induce the same group homomorphism $\pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Die Übungsblätter können zu zweit eingereicht werden. Abgabe der Übungsblätter im entsprechenden Fach im Keller des mathematischen Instituts.